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The Rainich Conditions for Neutrino Fields

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Abstract

The algebraic Rainich conditions for neutrino fields are obtained in terms of an algebraic classification of symmetric second-rank tensors. The differential Rainich conditions are also obtained for all but one class of neutrino fields. The algebraic classification of symmetric second-rank tensors used here is compared to two previously published classifications. The 'Weyl square' of a symmetric second-rank tensor is introduced and the Petrov type of the 'Weyl square' is used to give a method for determining the class of the symmetric tensor.

1. *Introduction*

In general relativity the geometry of the space-time is related to the source of the gravitational field through the field equation.

$$G_{ij} = -kT_{ij} \tag{1.1}$$

where G_{ij} is the Einstein tensor of the space-time and T_{ij} is the energy momentum tensor of the source. The Einstein tensor is given in terms of the Ricci tensor R_{ij} and the metric tensor g_{ij} by the equation

$$G_{ij} = R_{ij} - \tfrac{1}{2}Rg_{ij}$$

The sign conventions used here are fixed by the equations

$$v_{i;jk} - v_{i;kj} = R^l{}_{ijk}v_l \qquad \text{and} \qquad R_{ij} = R^l{}_{ijl}$$

The constant k appearing in (1.1) is then positive. It is assumed, on physical grounds, that the energy density relative to any observer is positive, that is

$$T_{ij}v^i v^j \geqslant 0 \tag{1.2}$$

for all time like vectors v^i.

The Rainich conditions for a given source are the necessary and sufficient conditions which must be imposed on the geometry of the space-time in order that the equation (1.1) be satisfied with T_{ij} the energy momentum tensor of the given source. In the case of a non-null electromagnetic field the Rainich conditions, obtained originally by Rainich (1925), can be written in terms of the metric tensor, the Ricci tensor and its covariant derivative. The null electromagnetic field is far more complicated and has been discussed by several authors including Ludwig (1970). The Rainich conditions for other sources have been found and usually consist of a set of algebraic conditions on the Ricci tensor together with a set of differential conditions.

As an example of the algebraic conditions consider a complex massless scalar field for which

$$R_{ij} = k(A_i A_j + B_i B_j) \tag{1.3}$$

Penney (1966) has obtained the algebraic Rainich conditions for this source in the form of an equation involving a sum of contractions of the product of three Ricci tensors. Since (1.3) does not involve the metric explicity it is also possible (and aesthetically desirable) to write the algebraic Rainich conditions in a form which does not involve contractions, namely,

$$[R_{kl} R_{pq} - R_{k(p} R_{q)l}] R_{ij} = R_{i(k} R_{l)j} R_{pq} + R_{i(p} R_{q)j} R_{kl} - R_{i(k} R_{l)(p} R_{q)j} - R_{j(k} R_{l)(p} R_{q)i}$$

where the round brackets denote symmetrisation. This example serves to illustrate that there is no unique way of writing the algebraic Rainich conditions as a set of tensor equations or inequalities. The canonical way to state algebraic Rainich conditions is rather to list the allowed *orbits* for the Ricci tensor. (By definition, two Ricci tensors lie on the same $\mathscr{L}_+^\uparrow$-orbit if and only if one is the transform of the other by a proper orthochronous Lorentz transfoımation.)

In Section 2, following Shaw (1971), we exhibit all possible orbits in the space of second-rank symmetric tensors. The Rainich conditions for a neutrino field are then discussed in Sections 3 and 4. Finally, in Section 5, a novel method for determining the algebraic class of a given tensor is developed.

2. *An Algebraic Classification of Symmetric Second-Rank Tensors*

The algebraic classification of symmetric second-rank tensors A_{ij} used here is best introduced by considering the tensor as a self adjoint linear mapping of Minkowski space

$$A: M \to M$$

The following five classes can then be distinguished:

Class D1 consists of those A which are diagonisable over $\mathbb{R}$.

Class D2 consists of those A which are diagonisable over $\mathbb{C}$ but not over $\mathbb{R}$.

Class N1$\pm$ consists of those A which leave invariant a decomposition $M = M_2 \perp E_2$, but whose restrictions to the space M_2 are non-diagonisable (even over $\mathbb{C}$); here the two two-spaces M_2 and E_2 are time-like and space-like respectively.

Class N2 consists of those A which do not possess an invariant non-singular two-space.

The corresponding canonical forms for A_{ij} are given in the following table.

TABLE 1. Canonical forms for A_{ij}

Class	Canonical form
D1	$-\mu_1 x_i x_j - \mu_2 y_i y_j - \mu_3 z_i z_j - \mu_4 t_i t_j, \quad \mu_1 \geqslant \mu_2 \geqslant \mu_3$
D2	$\mu(k_i l_j + l_i k_j) + \beta(l_i l_j - k_i k_j) - \mu_1 x_i x_j - \mu_2 y_i y_j, \quad \mu_1 \geqslant \mu_2, \beta > 0$
N1±	$\mu(k_i l_j + l_i k_j) \pm k_i k_j - \mu_1 x_i x_j - \mu_2 y_i y_j, \quad \mu_1 \geqslant \mu_2$
N2	$\mu(k_i l_j + l_i k_j - x_i x_j) - \mu_2 y_i y_j - k_i x_j - x_i k_j$

Here the vectors x_i, y_i, z_i, t_i form an orthonormal basis whilst x_i, y_i, k_i, l_i form a 'hybrid basis', that is, locally,

$$g_{ij} = -x_i x_j - y_i y_j - z_i z_j + t_i t_j \quad \text{or} \quad g_{ij} = -x_i x_j - y_i y_j + k_i l_j + l_i k_j$$

This classification of symmetric second-rank tensors is complete in the sense that each such tensor belongs to one and only one class and two such tensors are related by a proper orthochronous Lorentz transformation if and only if they belong to the same class and have identical canonical forms (i.e. the same parameters, *satisfying the inequalities stated in Table 1*). In other words the parameters serve to label the distinct $\mathscr{L}_+^\uparrow$-orbits. For example, the class N1+ splits up into distinct orbits (N1+) (μ, μ_1, μ_2) parameterised by the ordered triple of real numbers (μ, μ_1, μ_2) which are restricted in value only by the condition $\mu_1 \geqslant \mu_2$. Incidentally, it is easy to see that the $\mathscr{L}$-orbits coincide with the $\mathscr{L}_+^\uparrow$-orbits.

Previous classifications of symmetric second-rank tensors, for example those of Churchill (1932) and Plebanski (1964) do not exhibit clearly a list of all possible orbits.

The five classes given above can be subdivided in several ways. One method is to divide each of the five classes into 'types' by taking into account coincidences amongst the eigenvalues of A. Seventeen distinct types are then obtained:

D1: [1111], [211], [22], [31], [4];
D2: [11(1$\bar{1}$)], [2(1$\bar{1}$)];
N1±: [211], [22], [31], [4];
N2: [31], [4].

The symbol (1$\bar{1}$) denotes a pair of complex conjugate eigenvalues and so occurs only in the case of class D2, the types D2[11(1$\bar{1}$)] and D2[2(1$\bar{1}$)] corresponding to the two possibilities $\mu_1 \neq \mu_2$ and $\mu_1 = \mu_2$. Further details of this classification are given by Shaw (1971). The relationship between this classification and the two classifications mentioned above are summarised in the Appendix.

Because of the positive energy density condition mentioned in the introduction the following theorem is found to be useful later.

Theorem 1

If $A_{ij}v^i v^j \leqslant 0$ for all timelike vectors v^i then either $A_{ij} \in \mathrm{D1}$ with $\mu_4 \leqslant \mu_3$ and $\mu_4 \leqslant 0$ or $A_{ij} \in \mathrm{N1}-$ with $\mu \leqslant \mu_2$ and $\mu \leqslant 0$.

Proof

Suppose $A_{ij} \in \mathrm{D1}$. Then, using the notation $v_x = x_i v^i$ etc., the hypothesis of the theorem is that

$$-\mu_1 v_x^2 - \mu_2 v_y^2 - \mu_3 v_z^2 + \mu_4 v_t^2 \leqslant 0$$

for all v_x, v_y, v_z, v_t satisfying

$$-v_x^2 - v_y^2 - v_z^2 + v_t^2 > 0$$

Rewriting the first inequality as

$$(\mu_4 - \mu_1) v_x^2 + (\mu_4 - \mu_2) v_y^2 + (\mu_4 - \mu_3) v_z^2 + \mu_4(-v_x^2 - v_y^2 - v_z^2 + v_t^2) \leqslant 0$$

yields the result for $A_{ij} \in \mathrm{D1}$ immediately. All other classes can be dealt with in a similar manner.

3. *The Algebraic Rainich Conditions for Neutrino Fields*

The theory of neutrino fields in curved space-times is discussed at length by Brill & Wheeler (1957). Such a neutrino field is described by a two-component spinor ψ^A satisfying Weyl's equation

$$\sigma^{iAB}\psi_{B;i} = 0 \tag{3.1}$$

and the energy momentum tensor of the field is

$$T_{ij} = i[\sigma_i{}^{AB}(\bar{\psi}_{A;j}\psi_B - \bar{\psi}_A\psi_{B;j}) + \sigma_j{}^{AB}(\bar{\psi}_{A;i}\psi_B - \bar{\psi}_A\psi_{B;i})] \tag{3.2}$$

where $\sigma_i{}^{AB}$ are the generalised Pauli matrices and the semicolon denotes covariant differentiation. It follows from (3.1) and (3.2) that the trace of the energy momentum tensor vanishes and so the field equation (1.1) becomes

$$R_{ij} = -kT_{ij} \tag{3.3}$$

and

$$R = 0 \tag{3.4}$$

Using the notation and conventions of Penrose (1960) a trace-free Ricci tensor corresponds to a spinor $\phi_{AB\dot{C}\dot{D}}$ satisfying the reality condition

$$\phi_{AB\dot{C}\dot{D}} = \bar{\phi}_{\dot{C}\dot{D}AB}$$

In terms of this spinor, equation (3.3) with (3.2) substituted into the right-hand side, becomes

$$\phi_{AB\dot{C}\dot{D}} = i(\psi_A \partial_{B\dot{D}} \bar{\psi}_{\dot{C}} + \psi_B \partial_{A\dot{D}} \bar{\psi}_{\dot{C}} - \bar{\psi}_{\dot{C}} \partial_{B\dot{D}} \psi_A - \bar{\psi}_{\dot{D}} \partial_{B\dot{C}} \psi_A) \tag{3.5}$$

To solve the algebraic Rainich problem it is necessary to investigate which orbits of (trace-free) tensors R_{ij} correspond to a spinor which can be written in the form

$$\phi_{AB\dot{C}\dot{D}} = i(\psi_A \bar{\chi}_{\dot{D}B\dot{C}} + \psi_B \bar{\chi}_{\dot{D}A\dot{C}} - \bar{\psi}_{\dot{C}} \chi_{B\dot{D}A} - \bar{\psi}_{\dot{D}} \chi_{B\dot{C}A}) \tag{3.6}$$

The condition $\chi_{B\dot{C}A} \equiv \partial_{B\dot{C}} \psi_A$ forms part of the differential Rainich conditions.

Two theorems are now proved.

Theorem 2

The spinor $\phi_{AB\dot{C}\dot{D}}$ can be written in the form (3.6) if and only if a real null vector p^i ($\neq 0$) exists such that $R_{ij} p^i p^j = 0$.

Proof

Suppose the spinor $\phi_{AB\dot{C}\dot{D}}$ can be written in the form (3.6). Then $\phi_{AB\dot{C}\dot{D}} \psi^A \psi^B \bar{\psi}^{\dot{C}} \bar{\psi}^{\dot{D}} = 0$ so that

$$R_{ij} p^i p^j = 0$$

where p^i is the null vector corresponding to the spinor $\psi^A \bar{\psi}^{\dot{C}}$. Now suppose $R_{ij} p^i p^j = 0$. Introduce two basis spinors o_A, i_A so that p^i corresponds to $o^A \bar{o}^{\dot{C}}$. Then, using the notation of Newman & Penrose (1962), $\phi_{00} = 0$ and hence

$$\begin{aligned}\phi_{AB\dot{C}\dot{D}} = {} & \phi_{22}\, o_A o_B \bar{o}_{\dot{C}} \bar{o}_{\dot{D}} + \phi_{11}(i_A o_B + o_A i_B)(\bar{o}_{\dot{C}} \bar{\imath}_{\dot{D}} + \bar{\imath}_{\dot{C}} \bar{o}_{\dot{D}}) + \phi_{02}\, i_A i_B \bar{o}_{\dot{C}} \bar{o}_{\dot{D}} \\ & + \bar{\phi}_{02}\, o_A o_B \bar{\imath}_{\dot{C}} \bar{\imath}_{\dot{D}} - \phi_{01}\, i_A i_B (\bar{o}_{\dot{C}} \bar{\imath}_{\dot{D}} + \bar{\imath}_{\dot{C}} \bar{o}_{\dot{D}}) - \bar{\phi}_{01}(o_A i_B + i_A o_B) \bar{\imath}_{\dot{C}} \bar{\imath}_{\dot{D}} \\ & - \phi_{21}\, o_A o_B (\bar{o}_{\dot{C}} \bar{\imath}_{\dot{D}} + \bar{\imath}_{\dot{C}} \bar{o}_{\dot{D}}) - \bar{\phi}_{21}(i_A o_B + o_A i_B) \bar{o}_{\dot{C}} \bar{o}_{\dot{D}}\end{aligned}$$

It follows that $\phi_{AB\dot{C}\dot{D}}$ can be written in the form (3.6) with $\psi_A = o_A$ and

$$\begin{aligned}\chi_{B\dot{C}A} = i[& \tfrac{1}{4}\phi_{22}\, o_A o_B \bar{o}_{\dot{C}} + \tfrac{1}{2}\phi_{11}(o_A i_B + o_B i_A) \bar{\imath}_{\dot{C}} + \tfrac{1}{2}\phi_{02}\, i_A i_B \bar{o}_{\dot{C}} \\ & - \phi_{01}\, i_A i_B \bar{\imath}_{\dot{C}} - \tfrac{1}{2}\bar{\phi}_{21}(o_A i_B + o_B i_A) \bar{o}_{\dot{C}}]\end{aligned}$$

Theorem 3

If $A_{ij} p^i p^j = 0$ for some null vector then A_{ij} can be of any class excepting D1 with $\mu_4 > \mu_1$, or $\mu_4 < \mu_3$.

Proof

This theorem is proved in a similar manner to Theorem 1.

Combining the results of Theorems 1 and 3, and remembering that for a neutrino field the Ricci tensor is trace-free, the following Theorem is proved.

Theorem 4 (The algebraic Rainich conditions)

For a neutrino field with positive energy density either the Ricci tensor $R_{ij} \in$ D1 with $\mu_4 = \mu_3$ and $\mu_1 + \mu_2 + \mu_3 + \mu_4 = 0$, or $R_{ij} \in$ N1− with

$\mu \leqslant \mu_2$ and $2\mu + \mu_1 + \mu_2 = 0$. (If $R_{ij} \in$ D1 the condition $\mu_4 \leqslant 0$ for positive energy follows from $\mu_1 \geqslant \mu_2 \geqslant \mu_3$.)

In the electromagnetic case the corresponding conditions are well known: either $R_{ij} \in$ D1[22], with $\mu_1 = -\mu_4 \geqslant 0$, or $R_{ij} \in$ N1–[4], with $\mu = 0$.

Having obtained the algebraic Rainich conditions it is instructive to investigate how uniquely these conditions fix the neutrino field ψ_A. If $R_{ij} \in$ D1[211] or D1[22] with $\mu_4 = \mu_3$ then the null vector p_i is given by

$$p_i = \alpha l_i \qquad \text{or} \qquad p_i = \alpha k_i$$

If $R_{ij} \in$ D1[31], then

$$p_i = \alpha l_i + \beta k_i + \gamma y_i, \qquad \text{with } 2\alpha\beta = \delta^2$$

Finally if $R_{ij} \in$ N1–, then

$$p_i = \alpha k_i$$

In the above α, β, γ are all real. It follows that if $R_{ij} \in$ N1–, D1[211] or D1[22] then the neutrino field is given by

$$\psi_A = \epsilon^{i\theta} T o_A \tag{3.7}$$

where o_A is any spinor satisfying $k_i = \sigma_i{}^{A\dot{C}} o_A \bar{o}_{\dot{C}}$ (or possibly, for $R_{ij} \in$ D1[211] or D1[22], $l_i = \sigma_i{}^{A\dot{C}} o_A \bar{o}_{\dot{C}}$).

If $R_{ij} \in$ D1[31] then

$$\psi_A = \epsilon^{i\theta}(T o_A + S i_A) \tag{3.8}$$

where o_A and i_A are any two spinors satisfying

$$k_i = \sigma_i{}^{A\dot{C}} i_A \bar{i}_{\dot{C}}, \qquad l_i = \sigma_i{}^{A\dot{C}} o_A \bar{o}_{\dot{C}} \qquad \text{and} \qquad y_i = \sigma_i{}^{A\dot{C}}(o_A \bar{i}_{\dot{C}} + i_A \bar{o}_{\dot{C}})|\sqrt{2}$$

In the above T, S and θ are all real.

4. *The Differential Rainich Conditions for Neutrino Fields*

Substituting (3.7) into (3.1) and (3.5) yields the two equations

$$T\partial_{B\dot{D}} o_A - T\partial_{A\dot{D}} o_B + o_A[\partial_{B\dot{D}} T + iT\partial_{B\dot{D}}\theta] - o_B[\partial_{A\dot{D}} T + iT\partial_{A\dot{D}}\theta] = 0 \tag{4.1}$$

and

$$\phi_{AB\dot{C}\dot{D}} = T^2 \underset{0}{\phi}_{AB\dot{C}\dot{D}} + 2o_A \bar{o}_{\dot{C}} T^2 \partial_{B\dot{D}}\theta + io_B \bar{o}_{\dot{C}} T[\partial_{A\dot{D}} T - iT\partial_{A\dot{D}}\theta] - io_A \bar{o}_{\dot{D}} T[\partial_{B\dot{C}} T + iT\partial_{B\dot{C}}\theta] \tag{4.2}$$

where $\underset{0}{\phi}_{AB\dot{C}\dot{D}}$ is the spinor obtained by substituting o_A for ψ_A in (3.5). The differential Rainich conditions (for all neutrino fields but those belonging to D1[3 1]) are the conditions that these two equations admit a solution for T and θ. Contracting (4.2) with $o^A \bar{o}^{\dot{C}}$ yields

$$\phi_{AB\dot{C}\dot{D}} c^A \bar{o}^{\dot{C}} = T^2 \underset{0}{\phi}_{AB\dot{C}\dot{D}} o^A \bar{o}^{\dot{C}}$$

or, the tensor equivalent,

$$R_{ij}p^j = T^2 \underset{0}{R}_{ij}p^j \tag{4.3}$$

Now (4.3) yields an expression for T^2 except when $R_{ij}p^j = 0$. This condition implies that the neutrino field is a pure radiation field as discussed by Griffiths & Newing (1970) and Audretsch & Graf (1970). Since all such fields have been found explicitly by Collinson & Morris (1972) and by Trim & Wainwright (1971) it will be assumed now that $R_{ij}p^j \neq 0$. The condition for (4.3) to admit a solution for T^2 is that

$$p^j R_{j[i} \underset{0}{R}_{k]l} p^l = 0 \tag{4.4}$$

where the square brackets denote antisymmetrisation. If this equation is satisfied then T^2 is defined uniquely. Now introduce

$$\underset{1}{o}_A = T o_A$$

so that

$$\psi_A = \epsilon^{i\theta} \underset{1}{o}_A$$

Equation (4.2) can then be rewritten in the form

$$\phi_{AB\dot{C}\dot{D}} - \chi_{AB\dot{C}\dot{D}} = 4\underset{1}{o}_A \underset{1}{\bar{o}}_{\dot{C}} \partial_{B\dot{D}} \theta \tag{4.5}$$

where

$$\chi_{AB\dot{C}\dot{D}} = i[2\underset{1}{o}_A \partial_{B\dot{D}} \underset{1}{\bar{o}}_{\dot{C}} - 2\underset{1}{\bar{o}}_{\dot{C}} \partial_{B\dot{D}} \underset{1}{o}_A + \partial_{A\dot{D}}(\underset{1}{o}_B \underset{1}{\bar{o}}_{\dot{C}}) - \partial_{B\dot{C}}(\underset{1}{o}_A \underset{1}{\bar{o}}_{\dot{D}})]$$

The tensor form of (4.5) is

$$S_{ij} = 4\underset{1}{p}_i \theta_{;j} \tag{4.6}$$

where S_{ij} is the tensor corresponding to $\phi_{AB\dot{C}\dot{D}} - \chi_{AB\dot{C}\dot{D}}$. The equation (4.6) admits a solution for $\theta_{;j}$ if and only if the condition

$$\underset{1}{p}_{[k} S_{i]j} = 0 \tag{4.7}$$

is satisfied. As a differential equation for θ the following integrability condition must be satisfied

$$\underset{1}{p}_i S_{i[j;k]} = \underset{1}{p}_{i;[k|} S_{i|j]} \tag{4.8}$$

If (4.7) and (4.8) are satisfied then (4.6) defines θ up to an additive constant. The neutrino field is then defined up to a constant phase factor. The only remaining condition to be satisfied is the Weyl equation (4.1). After some manipulation this equation can be put in the form

$$\partial_{B\dot{D}}(\underset{1}{o}_A \underset{1}{\bar{o}}_{\dot{C}}) + \partial_{A\dot{C}}(\underset{1}{o}_B \underset{1}{\bar{o}}_{\dot{D}}) - \partial_{B\dot{C}}(\underset{1}{o}_A \underset{1}{\bar{o}}_{\dot{D}}) - \partial_{A\dot{D}}(\underset{1}{o}_B \underset{1}{\bar{o}}_{\dot{C}}) = 0 \tag{4.9}$$

The results of this section can now be summarised as:

Theorem 5 (The differential Rainich conditions)

Equations (4.4), (4.7), (4.8) and (4.9) constitute the differential Rainich conditions for a neutrino field (excluding only pure radiation fields and fields of class D1[31]). If $R_{ij} \in$ N1− the field is defined uniquely up to a constant phase factor. If $R_{ij} \in$ D1[211] or D1[22] then two distinct fields can exist, each defined up to a constant phase factor, one field derived from l_i the other from k_i.

The analysis for neutrino fields of class D1[31] is too difficult to complete at the present time. Other, unsuccessful approaches to the Rainich problem for neutrino fields have been made by Bergmann (1960), Penney (1965) and Inomata & McKinley (1965). The uniqueness of the pure radiation neutrino field has been discussed by Griffiths & Newing (1971) and by Collinson (1971).

5. *The Weyl Square*†

For any symmetric tensor A_{ij} the tensor $2A_{i[k}A_{l]j}$ has all the symmetries of the curvature tensor. From this tensor, a tensor A_{ijkl} can be constructed having all the algebraic properties of the Weyl tensor. A_{ijkl} is called the Weyl square of A_{ij} and is defined by the equation

$$A_{ijkl} = 2A_{i[k}A_{l]j} + g_{i[k}B_{l]j} - g_{j[k}B_{l]i} - \tfrac{1}{3}Bg_{i[k}g_{l]j} \qquad (5.1)$$

where

$$B_{ij} = B_{ji} = 2A^{i}{}_{[i}A_{i]j} \qquad \text{and} \qquad B = B^{j}{}_{j}$$

Using the Newman Penrose notation the null tetrad components of A_{ijkl} and of A_{ij} are related as in the following table.

TABLE 2. Null tetrad components of A_{ijkl} and of A_{ij}

A_{ijkl}	A_{ij}
ψ_0	$4(\phi_{01}^2 - \phi_{00}\phi_{02})$
ψ_1	$2(-\phi_{00}\phi_{12} + 2\phi_{01}\phi_{11} - \phi_{10}\phi_{02})$
ψ_2	$\tfrac{2}{3}(2\phi_{01}\phi_{11} - 4\phi_{12}\phi_{10} - \phi_{00}\phi_{22} + 4\phi_{11}^2 - \phi_{02}\phi_{20})$
ψ_3	$2(-\phi_{22}\phi_{10} + 2\phi_{21}\phi_{11} - \phi_{12}\phi_{20})$
ψ_4	$4(\phi_{21}^2 - \phi_{22}\phi_{20})$

It is remarkable that the trace of A_{ij} does not appear in the second column, that is the trace does not contribute to the Weyl square. This is

† The ideas discussed in this section have also been discussed by Plebanski (1964), although the authors were not aware of this fact when the work was being carried out. It is felt, nevertheless, that the approach used here might be found interesting.

rather analogous to the situation which occurs in the classification of Section 2 since any multiple of the metric can be added to A_{ij} without altering the class of the tensor. Substituting the canonical form for A_{ij} into the second column of Table 2 yields canonical forms for $\psi_0, \ldots, \psi_4$ which enable the Petrov type of the 'Weyl square' to be determined for each algebraic class of tensor A_{ij}. It turns out that knowledge of the Petrov type of the Weyl square and of the invariants

$$\sigma_2 = A^i{}_j A^j{}_i, \qquad \sigma_3 = A^i{}_j A^j{}_k A^k{}_i, \qquad \Delta = \det A^i{}_j$$

will determine the algebraic class of a (trace-free) tensor completely. The results are summarised in the following table.

TABLE 3. Relationship between the Petrov type of the Weyl square and the algebraic classification

Petrov type of A_{ijkl}	Further distinguishing conditions	Type of A_{ij}
I	$\tau^3 > 6(\tau')^2$	D1[1111]
	$\tau^3 < 6(\tau')^2$	D2[11(1$\bar{1}$)]
D	$\sigma > 0$	D1[211]
	$\sigma < 0$	D2[2(1$\bar{1}$)]
	$\sigma = 0$ and $A^i{}_j A^j{}_k = \frac{1}{4}\sigma_2\, \delta^i{}_k$	D1[22]
	$\sigma = 0$ and $A^i{}_j A^j{}_k \neq \frac{1}{4}\sigma_2\, \delta^i{}_k$	N1[22]
0	$\sigma_2 \neq 0$	D1[31]
	$\sigma_2 = 0$, $A_{ij} \neq 0$	N1[4]
	$A_{ij} = 0$	D1[4]
II		N1[211]
N	$\sigma_2 \neq 0$	N1[31]
	$\sigma_2 = 0$	N2[4]
III		N2[31]

In the above table

$$\sigma = \sigma_2\, \tau - 3\tau',$$

$$\tau = \tfrac{1}{6}\sigma_2^2 + 8\Delta,$$

and

$$\tau' = \tfrac{1}{3}\sigma_3^2 - \tfrac{1}{36}\sigma_2^3 + 4\sigma_2\, \Delta.$$

Details of the relationship between a second-order symmetric tensor and its Weyl square are given by Shaw (1971). It is possible that the results of Table 3 have more than curiosity value since the computer program developed by D'Inverno & Russell-Clark to determine the Petrov type of a given metric tensor could be modified to also determine the algebraic type of the Ricci tensor.

Appendix

The relationship between the classification introduced in Section 2 and the classifications of Churchill and Plebanski are summarised in the following tables

TABLE 4. Churchill's classification

Churchill	Section 2
I(a)	D1
I(b) or (c)	D2
I(d)	N1+ ∪ N1−
II	N2

TABLE 5. Plebanski's classification

Plebanski	Section 2
$[Z-\bar{Z}-S_1-S_2]_{[1-1-1-1]}$	D2[11(1$\bar{1}$)]
$[T-S_1-S_2-S_3]_{[1-1-1-1]}$	D1[1111]
$[2N-S_1-S_2]_{[2-1-1]}$	N1 + [211] ∪ N1 − [211]
$[3N-S]_{[3-1]}$	N2[31]
$[Z-\bar{Z}-2S]_{[1-1-1]}$	D2[2(1$\bar{1}$)]
$[2T-S_1-S_2]_{[1-1-1]}$	D1[211] with μ_1, μ_2, μ_3 all distinct
$[T-2S_1-S_2]_{[1-1-1]}$	D1[211] with not all μ_1, μ_2, μ_3 distinct
$[2N-2S]_{[2-1]}$	N1 + [22] ∪ N1 − [22]
$[3N-S]_{[2-1]}$	N1 + [31] ∪ N1 − [31]
$[4N]_{[3]}$	N2[4]
$[2T-2S]_{[1-1]}$	D1[22]
$[3T-S]_{[1-1]}$	D1[31] with μ_1, μ_2, μ_3 not all equal
$[T-3S]_{[1-1]}$	D1[31] with μ_1, μ_2, μ_3 all equal
$[4N]_{[2]}$	N1 + [4] ∪ N1 − [4]
$[4T]_{[1]}$	D1[4]

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